THE GROUPS OF STEINER IN PROBLEMS OF CONTACT*

BY

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1. The problems of contact discussed by STEINER† and HESSE† were investigated from a more general standpoint by CLEBSCH in his paper on the application of Abelian functions to geometry.‡ A study of the groups of these geometrical problems has been made by JORDAN.§ One of the most interesting of these groups was shown by JORDAN to be holoedrically isomorphic with the first hypoabelian linear group, which plays so important a rôle in various geometrical questions and in the problem of the construction of all solvable groups. As the proof (*Traité*, pp. 229–249) is quite complicated, it seemed to the writer worth while to publish the elementary proof given below of the isomorphism in question. No use will be made of the JORDAN substitutions $[\alpha_1, \beta_1, \cdots, \alpha_p, \beta_p]$, neither the origin nor the interpretation of which is apparent.

2. The theorem that there are 28 bitangents to a curve of the fourth order has been generalized by CLEBSCH (l. c., § 8) as follows: Let C_n be a curve of order n having no double points and set $p = \frac{1}{2}(n-1)(n-2)$. *There are* $2^{p-1}(2^p-1)$ *curves of order* $n-3$ *having simple contact with* C_n *at* $\frac{1}{2}n(n-3)$ *points.* The determination of these curves depends upon an equation E of degree $R_p \equiv 2^{2p-1} - 2^{p-1}$, whose roots may be represented by the symbol $(x_1 y_1 \cdots x_p y_p)$, where $x_1, y_1, \cdots, x_p, y_p$ may be 0 or 1, such that

$$(1) \qquad x_1 y_1 + x_2 y_2 + \cdots + x_p y_p \equiv 1 \pmod 2.$$

Let μ be any integer, $\mu \leqq R_p$, such that $\mu(n-3)/2$ is also an integer, and consider the μ roots

$$(x'_1 y'_1 \cdots x'_p y'_p), \cdots, (x_1^{(\mu)} y_1^{(\mu)} \cdots x_p^{(\mu)} y_p^{(\mu)}).$$

CLEBSCH proved that the points of contact of C_n with the corresponding μ

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†Journal für Mathematik, vol. 49 (1855).

‡Ibid., vol. 63 (1864), pp. 189–243.

§ *Traité des substitutions*, pp. 329–333, 305–308, 229–249.

curves all lie on a curve of order $\mu(n-3)/2$, if the following congruences hold simultaneously:

(2) $x'_i + x''_i + \cdots + x_i^{(\mu)} \equiv 0, \quad y'_i + y''_i + \cdots + y_i^{(\mu)} \equiv 0 \pmod 2 \quad (i=1, \cdots, p).$

Let ϕ_μ denote the sum of the products of the R_p roots taken μ at a time. According to a general principle,* the substitutions of the group G of the equation E will leave the function ϕ_μ invariant. If n be even, μ can have only even values, so that G is a subgroup of the group† which leaves $\phi_4, \phi_6, \cdots$ invariant. If n be odd, μ can be any integer such that $2 < \mu \leqq R_p$, and the group G is contained in the group G'_1 defined by the invariants $\phi_3, \phi_4, \cdots, \phi_{R_p}$. We are to prove that G'_1 is holoedrically isomorphic with the first hypoabelian group G_0 on $2p$ indices with coefficients taken modulo 2.

3. The first hypoabelian group G_0 is formed by the substitutions

$$S: \qquad \xi'_i = \sum_{j=1}^{p} (\alpha_{ij}\xi_j + \gamma_{ij}\eta_j), \quad \eta'_i = \sum_{j=1}^{p} (\beta_{ij}\xi_j + \delta_{ij}\eta_j) \quad (i=1, \cdots, p),$$

with coefficients taken modulo 2, which leave formally invariant the function

$$\theta \equiv \xi_1\eta_1 + \xi_2\eta_2 + \cdots + \xi_p\eta_p.$$

As generators of G_0, we may take

(3) $$M_i \equiv (\xi_i\eta_i), \quad N_{ij}: \ \xi'_i = \xi_i + \eta_j, \quad \xi'_j = \xi_j + \eta_i,$$

where we have written only the indices altered by the substitution.

The substitution S replaces the function

$$f = \sum_{i=1}^{p} (x_i\xi_i + y_i\eta_i)$$

by

$$f' = \sum_{i=1}^{p} (x'_i\xi_i + y'_i\eta_i), \quad x'_i \equiv \sum_{j=1}^{p} (\alpha_{ji}x_j + \beta_{ji}y_j), \quad y'_i \equiv \sum_{j=1}^{p} (\gamma_{ji}x_j + \delta_{ji}y_j).$$

The x'_i, y'_i are expressed in terms of x_j, y_j by formulæ which define a matrix of coefficients identical with the transposed of the matrix of the coefficients of S. Hence these formulæ define a substitution of the group G_0 (as shown by the explicit conditions on the coefficients of a first hypoabelian substitution).‡ Hence

(4) $$x'_1y'_1 + x'_2y'_2 + \cdots + x'_py'_p = x_1y_1 + x_2y_2 + \cdots + x_py_p.$$

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*Compare JORDAN, *Traité*, no. 421.

†Shown by JORDAN, nos. 319–335, to be holoedrically isomorphic with the Abelian linear group on $2p$ indices with coefficients taken modulo 2.

‡Cf. Bulletin of the American Mathematical Society, vol. 4 (1898), pp. 495–510.

This result may also be shown by considering the generators (3). In fact, M_1 and N_{12} replace the function f by, respectively,

$$y_1\xi_1 + x_1\eta_1 + \sum_{i=2}^{p} (x_i\xi_i + y_i\eta_i),$$

$$x_1\xi_1 + (y_1 + x_2)\eta_1 + x_2\xi_2 + (y_2 + x_1)\eta_2 + \sum_{j=3}^{p} (x_i\xi_i + y_i\eta_i).$$

In view of (4), it follows that S permutes amongst themselves the functions f in which $x_1y_1 + \cdots + x_py_p = 1$. In place of the functions f, we may employ the positional symbols $(x_1y_1 \cdots x_py_p)$ of § 2. Hence G_0 is isomorphic with a substitution-group Γ on these R_p symbols. Moreover, the isomorphism is holoedric and the group Γ is transitive; these results are readily proved.*

4. We may write the functions ϕ_3 and ϕ_4 as follows:

$$\phi_3 = \sum (x_1'y_1' \cdots x_p'y_p')(x_1''y_1'' \cdots x_p''y_p'')(x_1' + x_1''\ y_1' + y_1'' \cdots x_p' + x_p''\ y_p' + y_p''),$$
$$\phi_4 = \sum (x_1'y_1' \cdots)(x_1''y_1'' \cdots)(x_1'''y_1''' \cdots)(x_1' + x_1'' + x_1'''\ y_1' + y_1'' + y_1''' \cdots),$$

the summations extending over all the symbols $(x_1'y_1' \cdots)$, $(x_1''y_1'' \cdots)$, $(x_1'''y_1''' \cdots)$, such that the final term is, in each case, a symbol. Thus, for ϕ_3,

$$\sum_{i=1}^{p} x_i'y_i' \equiv 1, \quad \sum_{i=1}^{p} x_i''y_i'' \equiv 1, \quad \sum_{i=1}^{p} (x_i' + x_i'')(y_i' + y_i'') \equiv 1 \pmod 2.$$

Let G_1 be the group of STEINER composed of all substitutions on the R_p symbols $(x_1y_1 \cdots x_py_p)$ which leave ϕ_3 and ϕ_4 invariant.† We first show that G_1 contains the group Γ as a subgroup. In fact, M_1 replaces the general term (written above) of ϕ_3 by

$$(y_1'x_1'\ x_2'y_2' \cdots)(y_1''x_1''\ x_2''y_2'' \cdots)(y_1' + y_1''\ x_1' + x_1''\ x_2' + x_2''\ y_2' + y_2'' \cdots),$$

which is also a term of ϕ_3. Similarly, M_i and N_{ij} leave ϕ_3 and ϕ_4 invariant. Hence G_1 contains all the generators of Γ. The next step consists in the proof that every substitution of G_1 belongs to Γ. From the two results we may then conclude that $G_1 \equiv \Gamma$, so that G_1 and the first hypoabelian group G_0 will be proved holoedrically isomorphic.

5. Let L be an arbitrary substitution of G_1 and denote by f_1 the symbol which L replaces by (00 11 00 ⋯ 00). Then Γ, being transitive, contains a substitution L' which replaces f_1 by (00 11 ⋯ 00). Hence $M \equiv L'^{-1}L$ will belong to G_1 and will leave (00 11 ⋯ 00) fixed. Since M does not alter ϕ_3, it

* American Journal of Mathematics, vol. 23 (1901), pp. 337–377, § 26.

† It appears in the sequel that $G_1 \equiv G_1'$, the latter (§2) leaving ϕ_3, ϕ_4, $\cdots$, ϕ_{R_p} invariant.

will leave invariant the function ϕ_3' given by the sum of those terms in ϕ_3 which contain the factor (00 11 $\cdots$ 00):

$$\phi_3' \equiv \sum (00\ 11\ 00\ \cdots)(x_1y_1\ x_2y_2\ x_3y_3\cdots)(x_1y_1\ x_2+1\,y_2+1\ x_3y_3\cdots),$$

$$(5)\qquad \sum_{i=1}^{p} x_iy_i \equiv 1,\quad \sum_{i=1}^{p}{}' x_iy_i + (x_2+1)(y_2+1) \equiv 1 \pmod 2,$$

where the accent denotes that the value $i = 2$ is excluded. Hence $x_2 + y_2 \equiv 1$ (mod 2). Note that every set of solutions of (5) makes the three symbols in every triple of ϕ_3' all different. Hence M leaves invariant

$$\psi \equiv \sum (x_1y_1\ x_2y_2\ x_3y_3\cdots)(x_1y_1\ x_2+1\,y_2+1\ x_3y_3\cdots).$$

6. Hence M permutes amongst themselves the N symbols * *not* contained in the function ψ and different from (00 11 00 $\cdots$), namely, the symbols $(x_1y_1\ x_2y_2\cdots)$ for which, in contrast to (5),

$$(6)\qquad \sum_{i=1}^{p} x_iy_i \equiv 1,\quad \sum_{i=1}^{p} x_iy_i + x_2 + y_2 + 1 \equiv 0 \pmod 2.$$

Hence $x_2 + y_2 \equiv 0$ (mod 2). We next prove that the substitutions of Γ which leave fixed (00 11 $\cdots$) permute transitively the N symbols defined by (6). Among them occurs (10 11 00 $\cdots$ 00). We are to prove that Γ contains a substitution Σ leaving fixed the symbol (00 11 $\cdots$ 00) and replacing (10 11 $\cdots$ 00) by an arbitrary symbol $(x_1y_1\cdots x_py_p)$ in which

$$(6')\qquad \sum_{i=1}^{p} x_iy_i \equiv 1,\quad x_2 \equiv y_2 \pmod 2.$$

In view of § 3, we may think of the literal substitutions of Γ as linear hypoabelian substitutions on $\xi_1, \eta_1, \cdots \xi_p, \eta_p$. We are therefore to prove that there exists a first hypoabelian substitution S which leaves $\xi_2 + \eta_2$ fixed and replaces $\xi_1 + \xi_2 + \eta_2$ by

$$\sum_{i=1}^{p} (x_i\xi_i + y_i\eta_i),$$

subject to (6′). Hence S must leave $\xi_2 + \eta_2$ fixed and replace ξ_1 by

$$\sum_{i=1}^{p} (x_i\xi_i + y_i\eta_i) - \xi_2 - \eta_2,\quad \sum_{i=1}^{p}{}' x_iy_i + (x_2-1)(y_2-1) \equiv 0,\quad x_2 \equiv y_2 \pmod 2.$$

* The number $N = R_p - 2R_{p-1} - 1 \equiv 2^{2p-2} - 1$. Indeed, the number of sets of solutions of (5) equals the number of sets of solutions of

$$\sum_{i=1}^{p}{}' x_iy_i \equiv 1,$$

since $(x_2+1)x_2$ is even, which number is evidently R_{p-1}.

7. Changing the notation, we are to prove that G_0 contains a substitution S_1 which leaves $\xi_1 + \eta_1$ fixed and replaces ξ_2 by

$$\sum_{i=1}^{p}(X_i\xi_i + Y_i\eta_i), \quad \sum_{i=1}^{p} X_iY_i \equiv 0, \quad X_1 \equiv Y_1 \pmod 2.$$

If $X_1 \equiv 0$, we take as S_1 a substitution * which leaves ξ_1 and η_1 fixed and replaces ξ_2 by

$$\sum_{i=2}^{p}(X_i\xi_i + Y_i\eta_i), \quad \sum_{i=2}^{p} X_iY_i \equiv 0 \pmod 2.$$

If $X_1 \not\equiv 0 \pmod 2$, then $X_2, Y_2, \cdots, X_p, Y_p$ are not all zero. Applying a suitable transformation on $\xi_2, \cdots, \xi_p$, we may suppose that $X_2 \neq 0$. Now G_0 contains the following substitution leaving $\xi_1 + \eta_1$ invariant:

$$W \equiv Q_{21a}N_{12a}: \begin{cases} \xi_1' = \xi_1 + a\eta_2, \quad \eta_1' = \eta_1 + a\eta_2, \\ \xi_2' = \xi_2 + a\xi_1 + a\eta_1 + a^2\eta_2, \quad \eta_2' = \eta_2. \end{cases}$$

Also G_0 contains the substitution V which leaves ξ_1 and η_1 fixed and replaces ξ_2 by

$$X_2\xi_2 + (Y_2 + X_1^2/X_2)\eta_2 + \sum_{i=3}^{p}(X_i\xi_i + Y_i\eta_i),$$

since, in view of $X_1 = Y_1$,

$$X_2(Y_2 + X_1^2/X_2) + \sum_{i=3}^{p} X_iY_i = X_1^2 + \sum_{i=2}^{p} X_iY_i = \sum_{i=1}^{p} X_iY_i = 0.$$

If we take $a = X_1/X_2$, the required substitution S_1 is the product WV.

8. It follows that $M = \Sigma P$, where P is a substitution of G_1 which leaves fixed the symbols (00 11 00 $\cdots$ 00) and (10 11 00 $\cdots$ 00). Hence P leaves invariant the two functions ϕ_3' and ϕ_3'' formed respectively by those terms of ϕ_3 which contain (00 11 $\cdots$) and (10 11 $\cdots$) as a factor. Hence P leaves invariant the function ψ of § 5 derived from ϕ_3', and the following function derived from ϕ_3'':

$$\psi_1 \equiv \Sigma(x_1y_1\, x_2y_2\, x_3y_3 \cdots)(x_1 + 1\, y_1\, x_2 + 1\, y_2 + 1\, x_3y_3 \cdots).$$

Hence P will permute amongst themselves the symbols occurring in ψ_1 and not in ψ. These symbols $(x_1y_1 \cdots)$ are defined by

$$\sum_{i=1}^{p} x_iy_i \equiv 1, \quad \sum_{i=1}^{p}{}' x_iy_i + (x_2+1)(y_2+1) \equiv 0,$$

$$(x_1+1)y_1 + (x_2+1)(y_2+1) + \sum_{i=3}^{p} x_iy_i \equiv 1 \pmod 2.$$

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* Bulletin, l. c., p. 498.

Hence there are 2^{2p-3} such symbols satisfying the conditions

$$(7) \qquad \sum_{i=1}^{p} x_i y_i \equiv 1, \; x_2 \equiv y_2, \; y_1 \equiv 1 \pmod 2.$$

Among them occurs (01 11 00 $\cdots$ 00). We next prove that Γ contains a substitution T which leaves fixed (00 11 $\cdots$), (10 11 $\cdots$) and replaces (01 11 $\cdots$) by an arbitrary symbol $(x_1 y_1 \, x_2 y_2 \cdots)$ satisfying the conditions (7). We are to find a substitution T of the first hypoabelian group G_0 which leaves fixed $\xi_2 + \eta_2$ and $\xi_1 + \xi_2 + \eta_2$, but replaces $\eta_1 + \xi_2 + \eta_2$ by $x_1 \xi_1 + y_1 \eta_1 + x_2 \xi_2 + y_2 \eta_2 + \cdots$ subject to the relations (7). Then T must leave fixed ξ_1 and $\xi_2 + \eta_2$, but replace η_1 by

$$x_1 \xi_1 + \eta_1 + (x_2 + 1) \xi_2 + (x_2 + 1) \eta_2 + \sum_{i=3}^{p} (x_i \xi_i + y_i \eta_i) \qquad \left[x_1 + x_2 + \sum_{i=3}^{p} x_i y_i \equiv 1\right].$$

Such a substitution belonging to G_0 is the following:

	ξ_1	η_1	ξ_2	η_2	ξ_3	η_3		ξ_p	η_p
$\xi_1' =$	1	0	0	0	0	0	$\cdots$	0	0
$\eta_1' =$	x_1	1	x_2+1	x_2+1	x_3	y_3	$\cdots$	x_p	y_p
$\xi_2' =$	x_2+1	0	1	0	0	0	$\cdots$	0	0
$\eta_2' =$	x_2+1	0	0	1	0	0	$\cdots$	0	0
$\xi_3' =$	a_{31}	γ_{31}	a_{32}	γ_{32}	a_{33}	γ_{33}	$\cdots$	a_{3p}	γ_{3p}
.	.	.	.	.	.	.	.	.	.
$\eta_p' =$	β_{p1}	δ_{p1}	β_{p2}	δ_{p2}	β_{p3}	δ_{p3}	$\cdots$	β_{pp}	δ_{pp}

Since the coefficients of the first four rows satisfy the first hypoabelian conditions which affect those rows, there exist values of

$$a_{ij}, \gamma_{ij}, \beta_{ij}, \delta_{ij} \qquad (i=3, \cdots, p; j=1, \cdots, p)$$

for the remaining $2p-4$ rows which give rise to a first hypoabelian substitution.*

9. It follows that $P = TQ$, where Q is a substitution of G_1 which leaves fixed the symbols (00 11 00 $\cdots$), (10 11 00 $\cdots$), and (01 11 00 $\cdots$). Since Q leaves ϕ_4 fixed, it will leave invariant the functions τ, τ_1, τ_2 which occur in ϕ_4 each multiplied by the respective factors (00 11 $\cdots$)(10 11 $\cdots$), (00 11 $\cdots$)(01 11 $\cdots$), (10 11 $\cdots$)(01 11 $\cdots$), namely,

$$\tau \equiv \sum (x_1 y_1 \, x_2 y_2 \cdots)(x_1 + 1 \, y_1 \, x_2 y_2 \cdots),$$
$$\tau_1 \equiv \sum (x_1 y_1 \, x_2 y_2 \cdots)(x_1 \, y_1 + 1 \, x_2 y_2 \cdots),$$
$$\tau_2 \equiv \sum (x_1 y_1 \, x_2 y_2 \cdots)(x_1 + 1 \, y_1 + 1 \, x_2 y_2 \cdots).$$

* *The successive generality theorem*, American Journal, l. c.

Hence Q will permute amongst themselves the q symbols which occur in τ and τ_1, but not in τ_2, subject therefore to the conditions

$$\sum_{i=1}^{p} x_i y_i \equiv 1, \quad (x_1+1)y_1 + \sum_{i=2}^{p} x_i y_i \equiv 1, \quad x_1(y_1+1) + \sum_{i=2}^{p} x_i y_i \equiv 1,$$

$$(x_1+1)(y_1+1) + \sum_{i=2}^{p} x_i y_i \equiv 0 \pmod 2.$$

Hence

$$x_1 \equiv y_1, \quad x_1 y_1 \equiv 0, \quad \sum_{i=2}^{p} x_i y_i \equiv 1 \pmod 2.$$

We obtain the q symbols *

$$(8) \qquad (00\, x_2 y_2\, x_3 y_3 \cdots), \quad \sum_{i=2}^{p} x_i y_i \equiv 1.$$

10. The theorem that $G_1 = \Gamma$ may now be proved by induction from $2(p-1)$ to $2p$ indices. We denote by $\phi_3^{(p-1)}$, $\phi_4^{(p-1)}$, $\cdots$ the functions composed of those terms of ϕ_3, ϕ_4, $\cdots$, respectively, which are formed exclusively of the symbols $(00\, x_2 y_2\, x_3 y_3 \cdots)$. We assume that every substitution which leaves fixed $\phi_3^{(p-1)}$, $\phi_4^{(p-1)}$ is derived from the substitutions of Γ_{p-1}, the first hypoabelian group on $p-1$ pairs of indices; and proceed to prove that every substitution which leaves fixed ϕ_3, ϕ_4 is derived from the substitutions of $\Gamma \equiv \Gamma_p$. In view of the earlier sections we need only consider the substitutions of the form Q which permute amongst themselves the q symbols (8). Let Q' be the substitution derived from Q by retaining only the cycles on the q symbols. Since Q' leaves $\phi_3^{(p-1)}$ and $\phi_4^{(p-1)}$ invariant, it belongs to Γ_{p-1} by hypothesis. We proceed to show that $K \equiv QQ'^{-1}$ reduces to the identity, so that the theorem will be proved. Now K leaves fixed every symbol $(00\, x_2 y_2 x_3 x_3 \cdots)$, as well as $(01\ 11\ 00 \cdots)$, and $(10\ 11\ 00 \cdots)$. Hence it leaves fixed the fourth term of ϕ_4 in the products

$$(00\ 11\ 00\ \cdots)(10\ 11\ 00\ \cdots)(00\ x_2 y_2\, x_3 y_3\ \cdots),$$

$$(00\ 11\ 00\ \cdots)(01\ 11\ 00\ \cdots)(00\ x_2 y_2\, x_3 y_3\ \cdots),$$

which are $(10\, x_2 y_2\, x_3 y_3 \cdots)$ and $(01\, x_2 y_2\, x_3 y_3 \cdots)$, respectively. Hence K leaves fixed the fourth term of ϕ_4 in the product

$$(10\ 1+x_2+x_3y_3\ 0\ 11\ 00\cdots)(01\ 0\ y_2+1\ 11\ 00\cdots)(00\ 1+x_3y_3\ 1\ x_3y_3\, x_4y_4\cdots),$$

which is $\sigma \equiv (11\, x_2 y_2\, x_3 y_3\, x_4 y_4 \cdots)$, where $x_4 y_4 + \cdots + x_p y_p \equiv 0$, $x_2 y_2 + x_3 y_3 \equiv 0$. But, in every symbol σ, $x_2 y_2 + x_3 y_3 + x_4 y_4 + \cdots + x_p y_p \equiv 0$. If there are any terms $\neq 0$, say $x_r y_r$ and $x_s y_s$, where $r > 1$, $s > 1$, $r \neq s$, then $x_r y_r + x_s y_s \equiv 0$

* Evidently, $q = R_{p-1} \equiv 2^{2p-3} - 2^{p-2}$.

(mod 2). Such a symbol σ may be reached in a manner analogous to that by which was obtained the σ having $x_2y_2 + x_3y_3 \equiv 0$.

Since K leaves fixed every symbol of the forms

$$(00\ x_2y_2\ x_3y_3 \cdots),\quad (10\ x_2y_2\ x_3y_3 \cdots),\quad (01\ x_2y_2\ x_3y_3 \cdots),\quad (11\ x_2y_2\ x_3y_3 \cdots),$$

it leaves every symbol fixed and is the identity.

11. The order Ω_p of the group $G_1 \equiv \Gamma$ may be derived from the preceding investigation. We have, for $p > 2$,

$$\Omega_p = R_p N 2^{2p-3}\Omega_{p-1} \div q \equiv (2^{2p-1} - 2^{p-1})(2^{2p-2} - 1)\, 2^{2p-3}\Omega_{p-1} \div (2^{2p-3} - 2^{p-2}),$$

upon substituting the values of R_p, N and q given in the notes to § 6 and § 9. The factor $\Omega_{p-1} \div q$ expresses the number of substitutions on the q symbols $(00\ x_2y_2\ x_3y_3 \cdots)$ which leave invariant the symbol $(00\ 11\ 00 \cdots)$. But Γ_{p-1} is transitive on these q symbols.

After simplification, we derive the recursion formula

$$\Omega_p = 2^{2p-2}(2^p - 1)(2^{p-1} + 1)\,\Omega_{p-1} \qquad (p > 2).$$

The formula holds also for $p = 2$, if we take $\Omega_1 = 2$, as must be done in the case of Γ, the hypoabelian substitutions on ξ_1 and η_1 being M_1 and the identity. The definition of G_1 for $p = 1$ is delusive since $R_1 = 1$; but, for $p = 2$, G_1 is formed of the $36 \equiv 2(3!)^2$ substitutions on $R_2 = 6$ symbols which leave invariant

$$\phi_3 \equiv (00\ 11)(11\ 01)(11\ 10) + (11\ 00)(10\ 11)(01\ 11).$$

We readily find that*

$$\Omega_p = (2^p - 1)\,[(2^{2p-2} - 1)\,2^{2p-2}]\,[(2^{2p-4} - 1)\,2^{2p-4}] \cdots [(2^2 - 1)\,2^2]\,2.$$

The factors of composition of Γ are known to be 2 and $\frac{1}{2}\Omega_p$, if $p > 2$.

12. The question of the generalization of the results of the paper from the field of integers taken modulo 2 to the Galois Field of order 2^m will be reserved for a later paper. It may be remarked that the results of §§ 3, 4, 5, and 7 are true for the $GF[2^m]$; but that the methods of §§ 6, 8, 9, and 10 would require essential modifications.

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* This result is in accord with that obtained otherwise in the Bulletin, l. c.